Q-FW: A Hybrid Classical-Quantum Frank-Wolfe for Quadratic Binary Optimization

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Introduction

Adiabatic Quantum Computers (AQCs) excel in solving **Q**uadratic **U**nconstrained **B**inary **O**ptimization Problems:

$$\arg\min_{\mathbf{x}\in\mathcal{B}} \mathbf{x}^\top \mathbf{Q}'\mathbf{x} \qquad \textbf{(QUBO)}$$

In most vision problems, constraints are required:

$$\min_{x\in\mathcal{B}} \ \mathbf{x}^\top \mathbf{Q}\mathbf{x} \quad \text{s.t.} \quad \mathbf{a}_i^\top \mathbf{x} = b_i, \quad i = 1\dots, m$$

This can be turned into a QUBO with ***regularization***. Yet, constraints are not **exactly** satisfied.

We introduce **Q-FW**, a true quantum-classical **hybrid** solver tailored for **binary** optimization problems subject to **linear equality and inequality constraints**.

Such problems occur frequently in computer vision. Our solver enables the use of quantum hardware for computer vision, paving the way to **quantum computer vision**.

Copositive Programming

Copositive **P**rogramming concerns optimization over the set of **completely positive matrices**:

$$\Delta := \operatorname{conv}\{\mathbf{w}\mathbf{w}^\top : \mathbf{w}\in\mathcal{B}\}$$

The standard form (**CP**) is:

$$\min_{\mathbf{W}\in\Delta} \ \operatorname{Tr}(\mathbf{C}\mathbf{W}) \quad \text{s.t.} \quad \mathcal{A}\mathbf{W} = \mathbf{v} \qquad \textbf{(CP)}$$

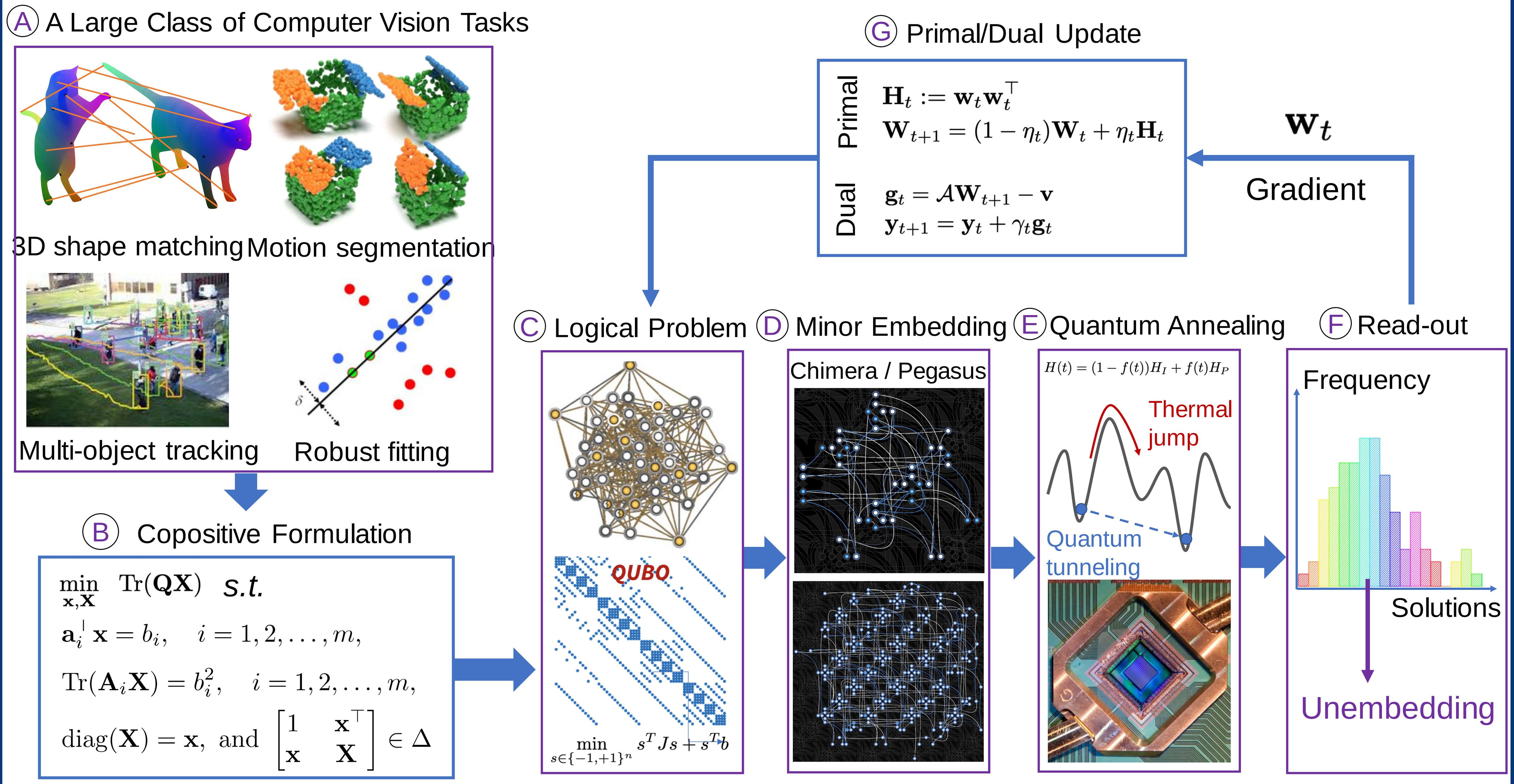
(**CP**) has the modeling power of non-convex optimization with the interpretability of convex optimization.

We reformulate (**QUBO**) with constraints as a (**CP**):

$$\min_{\mathbf{x},\mathbf{X}} \ \operatorname{Tr}(\mathbf{Q}\mathbf{X}) \quad \text{s.t.} \quad \mathbf{a}_i^\top \mathbf{x} = b_i, \quad i = 1,2,\dots,m,$$
$$\operatorname{Tr}(\mathbf{A}_i\mathbf{X}) = b_i^2, \quad i = 1,2,\dots,m,$$
$$\operatorname{diag}(\mathbf{X}) = \mathbf{x}, \ \text{and} \ \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X}\end{bmatrix} \in \Delta$$

This reformulation is tight.

Q-FW: A Hybrid Solver for Quantum Computer Vision



Quantum Frank Wolfe

Challenge : Solving (**CP**) is NP-Hard.
Opportunity : **L**inear **M**inimization over Δ is a (**QUBO**).

$$\mathbf{u} = \arg\min_{\mathbf{w}\in\mathcal{B}} \operatorname{Tr}(\mathbf{w}^\top\mathbf{C}\mathbf{w}) \iff \mathbf{u}\mathbf{u}^\top = \arg\min_{\mathbf{W}\in\Delta} \operatorname{Tr}(\mathbf{C}\mathbf{W}) \qquad \textbf{(LM)}$$

Idea: Design a Frank-Wolfe-type algorithm and solve (**LM**) step with an AQC.

Consider the **Augmented Lagrangian**: $L_\beta(\mathbf{W};\mathbf{y}) = \operatorname{Tr}(\mathbf{C}\mathbf{W}) + \mathbf{y}^\top(\mathcal{A}\mathbf{W} - \mathbf{v}) + \frac{\beta}{2}\|\mathcal{A}\mathbf{W} - \mathbf{v}\|^2 \quad \text{for } \mathbf{W}\in\Delta.$

Algorithm 1 Q-FW for Quadratic Binary Optimization.
Initialization: $\mathbf{W} \leftarrow \mathbf{0}, \ \mathbf{y} \leftarrow \mathbf{0}, \ \mathbf{g} \leftarrow -\mathbf{v}, \ \beta_0 > 0$ **Main loop**

[FWAL]:

for $t = 1,\dots,T$ **do**
- $\eta \leftarrow 2/(t+1)$, and $\beta \leftarrow \beta_0\sqrt{t+1}$
- $\mathbf{G} \leftarrow \mathbf{C} + \mathcal{A}^\top(\mathbf{y} + \beta\mathbf{g})$ — Primal step, inspired by FW
- $\mathbf{w} \leftarrow \arg\min\{\mathbf{w}^\top\mathbf{G}\mathbf{w} : \mathbf{w}\in\mathcal{B}\}$ ▷ QUBO subproblem
- $\mathbf{W} \leftarrow (1-\eta)\mathbf{W} + \eta\mathbf{w}\mathbf{w}^\top$
- $\mathbf{g} \leftarrow \mathcal{A}\mathbf{W} - \mathbf{v}$ — Dual step, gradient ascent
- $\mathbf{y} \leftarrow \mathbf{y} + \gamma\mathbf{g}$ ▷ In practice, we use $\gamma = \beta_0$

Rounding: Extract $\mathbf{X}$ by removing the first row and first column of $\mathbf{W}$. Compute $\mathbf{x}$ as the top singular vector of $\mathbf{X}$. – Project $\mathbf{x}$ onto $\mathcal{B}$.
Output: Solution $\mathbf{W}\in\Delta$ for (CP), and $\mathbf{x}\in\mathcal{B}$ for (QBO).

Theoretical guarantees:

$$\operatorname{Tr}(\mathbf{C}\mathbf{W}_t) - \operatorname{Tr}(\mathbf{C}\mathbf{W}_\star) \le \mathcal{O}(1/\sqrt{t})$$
$$\|\mathcal{A}\mathbf{W}_t - \mathbf{v}\| \le \mathcal{O}(1/\sqrt{t})$$

(**QUBO**) subproblem, solved at AQC

AQCs classically work for binary valued optimization problems. By integrating it with the FW algorithm, we lay the first stone for using AQCs on real-valued problems in CV and ML.

Evaluations

Graph Matching on Random Problems

$$\max_{\mathbf{\Pi}} \ \operatorname{vec}(\mathbf{\Pi})^\top \mathbf{Q}_{\text{QGM}} \operatorname{vec}(\mathbf{\Pi}) \quad \text{subject to} \quad \mathbf{\Pi}\in\mathcal{P}$$

N	[72] ins.	[72] row.	[7] DS*	[51] SA	FWAL	Q-FWAL (250)
3	1.49	2.12	0.85	0.82	**7e-4**	**7e-4**
4	5.68	7.37	0.43	2.43	**1.3e-3**	**1.43e-3**

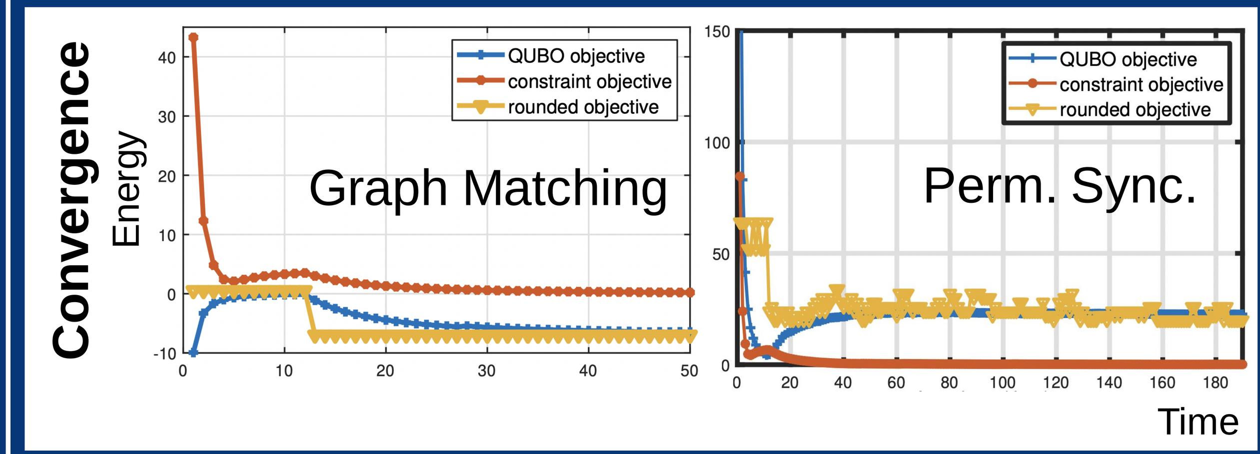
Table reports mean normalized energies over ten instances.

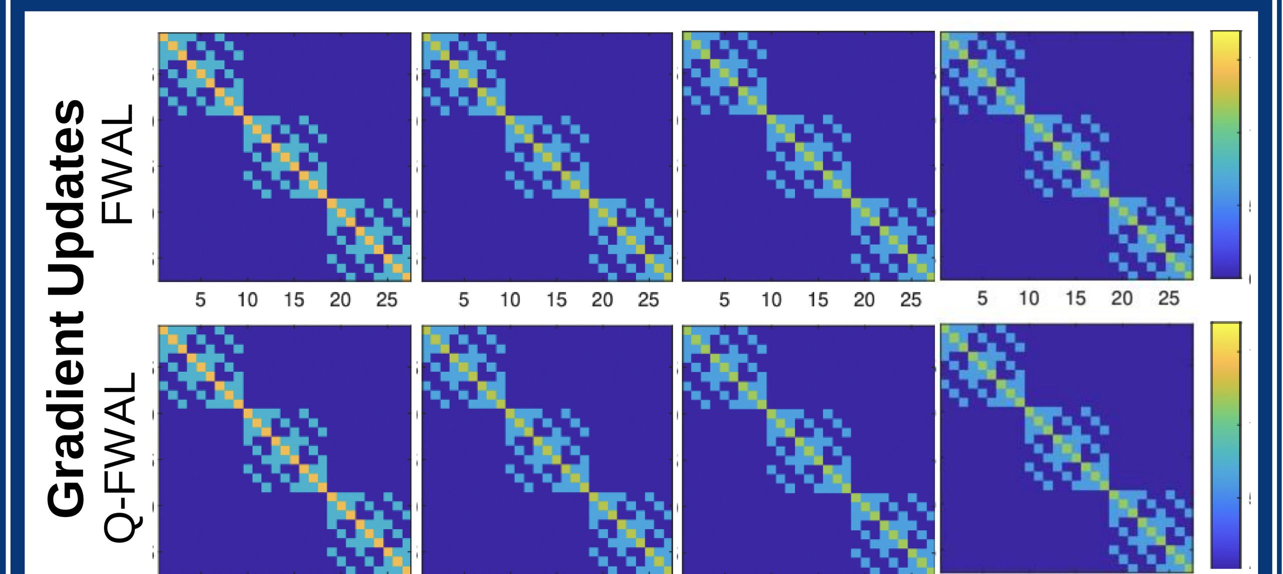
Permutation Synchronization

	Average
MatchEIG [57]	0.83 ± 0.088
MatchALSS [80]	0.87 ± 0.092
MatchLIFT [45]	0.87 ± 0.094
MatchBirkhoff [10]	0.87 ± 0.093
QuantumSync [9]	0.87 ± 0.096
[9]-search	0.88 ± 0.104
Q-FWAL (ours)	**0.94 ± 0.076**

Table reports accuracies as in [9].

$$\arg\min_{\{\mathbf{X}_i\in\mathcal{P}_n\}} \sum_{(i,j)\in\mathcal{E}} \|\mathbf{P}_{ij} - \mathbf{X}_i\mathbf{X}_j^\top\|_F^2 = \arg\min_{\{\mathbf{X}_i\in\mathcal{P}_n\}} \mathbf{x}^\top \mathbf{Q}_{QPS}\mathbf{x}$$





References

1. Birdal and Golyanik *et al.* "Quantum permutation synchronization." CVPR, 2021.
2. Yurtsever *et al.* "A conditional-gradient-based augmented Lagrangian framework". ICML, 2019.
3. Jaggi. "Revisiting Frank-Wolfe: Projection-free sparse convex optimization." ICML, 2013.
4. Benkner et al. "Adiabatic quantum graph matching with permutation matrix constraints." 3DV, 2020.